

Blow-up phenomena and global existence for the periodic two-component Dullin-Gottwald-Holm system

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Abstract

This paper is concerned with blow-up phenomena and global existence for the periodic two-component Dullin-Gottwald-Holm system. We first obtain several blow-up results and the blow-up rate of strong solutions to the system. We then present a global existence result for strong solutions to the system.

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1 Introduction

In this paper, we consider the following periodic two-component Dullin-Gottwald-Holm (DGH) system:

$$\left\{ \begin{array}{ll} m_t - Au_x + um_x + 2u_xm + \gamma u_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}, \end{array} \right. \quad (1.1)$$

where $m = u - u_{xx}$, $A > 0$ and γ are constants.

The system (1.1) has been recently derived by Zhu et al. in [1] by follow Ivanov's approach [2]. It was shown in [1] that the DGH system is completely integrable and can be written as a compatibility condition of two linear systems

$$\Psi_{xx} = \left(-\xi^2 \rho^2 + \xi \left(m - \frac{A}{2} + \frac{\gamma}{2} \right) + \frac{1}{4} \right) \Psi$$

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and

$$\Psi_t = \left(\frac{1}{2\xi} - u + \gamma \right) \Psi_x + \frac{1}{2} u_x \Psi,$$

where ξ is a spectral parameter. Moreover, this system has the following two Hamiltonians

$$E(u, \rho) = \frac{1}{2} \int (u^2 + u_x^2 + (\rho - 1)^2) dx$$

and

$$F(u, \rho) = \frac{1}{2} \int (u^3 + uu_x^2 - Au^2 - \gamma u_x^2 + 2u(\rho - 1) + u(\rho - 1)^2) dx.$$

For $\rho = 0$ and $m = u - \alpha^2 u_{xx}$, (1.1) becomes to the DGH equation [3]

$$u_t - \alpha^2 u_{txx} - Au_x + 3uu_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}),$$

where A and α are two positive constants, modeling unidirectional propagation of surface waves on a shallow layer of water which is at rest at infinity, $u(t, x)$ standing for fluid velocity. It is completely integrable with a bi-Hamiltonian and a Lax pair. Moreover, its traveling wave solutions include both the KdV solitons and the CH peakons as limiting cases [3]. The Cauchy problem of the DGH equation has been extensively studied, cf. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

For $\rho \neq 0$, $\gamma = 0$, the system (1.1) becomes to the two-component Camassa-Holm system [2]

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho \rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \quad (1.2)$$

where $\rho(t, x)$ in connection with the free surface elevation from scalar density (or equilibrium) and the parameter A characterizes a linear underlying shear flow. The system (1.2) describes water waves in the shallow water regime with nonzero constant vorticity, where the nonzero vorticity case indicates the presence of an underlying current. A large amount of literature was devoted to the Cauchy problem (1.2), see [17, 18, 19, 20, 21, 22, 23, 24, 25].

The Cauchy problem (1.1) has been discussed in [1]. Therein Zhu and Xu established the local well-posedness to the system (1.1), derived the precise blow-up scenario and investigated the wave breaking for the system (1.1). The aim of this paper is to study further the blow-up phenomena for strong solutions to (1.1) and to present a global existence result.

Our paper is organized as follows. In Section 2, we briefly give some needed results including the local well posedness of the system (1.1), the precise blow-up scenarios and some useful lemmas to study blow-up phenomena and global existence. In Section 3, we give several new blow-up results and the precise blow-up rate. In Section 4, we present a new global existence result of strong solutions to (1.1).

Notation Given a Banach space Z , we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over \mathbb{S} , for simplicity, we drop \mathbb{S} in our notations if there is no ambiguity.

2 Preliminaries

In this section, we will briefly give some needed results in order to pursue our goal. With $m = u - u_{xx}$, we can rewrite the system (1.1) as follows:

$$\begin{cases} u_t - u_{txx} - Au_x + \gamma u_{xxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}. \end{cases} \quad (2.1)$$

Note that if $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, $x \in \mathbb{R}$ is the kernel of $(1 - \partial_x^2)^{-1}$, then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{S})$, $G * m = u$. Here we denote by $*$ the convolution. Using this identity, we can rewrite the system (2.1) as follows:

$$\begin{cases} u_t + (u - \gamma)u_x = -\partial_x G * (u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2), & t > 0, x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}, \end{cases} \quad (2.2)$$

The local well-posedness of the Cauchy problem (2.1) can be obtained by applying the Kato's theorem. As a result, we have the following well-posedness result.

Lemma 2.1. ([1]). Given an initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, there exists a maximal $T = T(\|(u_0, \rho_0)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution

$$(u, \rho) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$$

of (2.1). Moreover, the solution (u, ρ) depends continuously on the initial data (u_0, ρ_0) and the maximal time of existence $T > 0$ is independent of s .

Consider now the following initial value problem

$$\begin{cases} q_t = u(t, q), & t \in [0, T], \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \quad (2.3)$$

where u denotes the first component of the solution (u, ρ) to (2.1).

Lemma 2.2. ([1]). Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$. Then Eq.(2.3) has a unique solution $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} with

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x))ds\right) > 0, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Lemma 2.3. ([1]). Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and $T > 0$ be the maximal existence. Then we have

$$\rho(t, q(t, x))q_x(t, x) = \rho_0(x), \quad (t, x) \in [0, T) \times \mathbb{S}.$$

Moreover, if there exists a $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, then $\rho(t, q(t, x_0)) = 0$ for all $t \in [0, T)$.

Next, we will give two useful conservation laws of strong solutions to (2.1).

Lemma 2.4. ([1]). Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and $T > 0$ be the maximal existence. Then for all $t \in [0, T)$, we have

$$\int_{\mathbb{S}} (u^2 + u_x^2 + \rho^2) dx = \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2 + \rho_0^2) dx := E_0.$$

Lemma 2.5. Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and $T > 0$ be the maximal existence. Then for all $t \in [0, T)$, we have

$$\int_{\mathbb{S}} u(t, x) dx = \int_{\mathbb{S}} u_0(x) dx.$$

Proof. By the first equation in (2.1), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}} u(t, x) dx &= \int_{\mathbb{S}} u_t dx \\ &= \int_{\mathbb{S}} (u_{txx} + Au_x - \gamma u_{xxx} - 3uu_x + 2u_x u_{xx} + uu_{xxx} - \rho \rho_x) dx = 0 \end{aligned}$$

This completes the proof of the lemma. \square

Then, we state the following precise blow-up mechanism of (2.1).

Lemma 2.6. ([1]). Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and $T > 0$ be the maximal existence. Then the solution blows up in finite time if and only if

$$\liminf_{t \rightarrow T^-} \left\{ \inf_{x \in \mathbb{S}} u_x(t, x) \right\} = -\infty.$$

Lemma 2.7. ([26]). Let $t_0 > 0$ and $v \in C^1([0, t_0]; H^2(\mathbb{R}))$. Then for every $t \in [0, t_0)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with

$$m(t) := \inf_{x \in \mathbb{R}} \{v_x(t, x)\} = v_x(t, \xi(t)),$$

and the function m is almost everywhere differentiable on $(0, t_0)$ with

$$\frac{d}{dt} m(t) = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, t_0).$$

Lemma 2.8. ([27]). (i) For every $f \in H^1(\mathbb{S})$, we have

$$\max_{x \in [0,1]} f^2(x) \leq \frac{e+1}{2(e-1)} \|f\|_{H^1}^2,$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

(ii) For every $f \in H^3(\mathbb{S})$, we have

$$\max_{x \in [0,1]} f^2(x) \leq c \|f\|_{H^1}^2,$$

with the best possible constant c lying within the range $(1, \frac{13}{12}]$. Moreover, the best constant c is $\frac{e+1}{2(e-1)}$.

Lemma 2.9. ([28]). If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x) dx = \frac{a_0}{2}$, then for every $\epsilon > 0$, we have

$$\max_{x \in [0,1]} f^2(x) \leq \frac{\epsilon+2}{24} \int_{\mathbb{S}} f_x^2 dx + \frac{\epsilon+2}{4\epsilon} a_0^2.$$

Moreover,

$$\max_{x \in [0,1]} f^2(x) \leq \frac{\epsilon+2}{24} \|f\|_{H^1(\mathbb{S})}^2 + \frac{\epsilon+2}{4\epsilon} a_0^2.$$

Lemma 2.10. ([29]). Assume that a differentiable function $y(t)$ satisfies

$$y'(t) \leq -Cy^2(t) + K \tag{2.4}$$

with constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution to (2.4) goes to $-\infty$ before t tend to $\frac{1}{-Cy_0 + \frac{K}{y_0}}$.

3 Blow-up phenomena

In this section, we discuss the blow-up phenomena of the system (2.1). Firstly, we prove that there exist strong solutions to (2.1) which do not exist globally in time.

Theorem 3.1. Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and T be the maximal time of the solution (u, ρ) to (2.1) with the initial data (u_0, ρ_0) . If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and

$$u'_0(x_0) = \inf_{x \in \mathbb{S}} u'_0(x) < -\sqrt{\frac{e+1}{2(e-1)} E_0 + |\gamma - A| \sqrt{\frac{8(e+1)}{e-1}} E_0^{\frac{1}{2}}},$$

then the corresponding solution to (2.1) blows up in finite time.

Proof. Applying Lemma 2.1 and a simple density argument, we only need to show that the above theorem holds for some $s \geq 2$. Here we assume $s = 3$ to prove the above theorem.

Define now

$$m(t) := \inf_{x \in \mathbb{S}} [u_x(t, x)], \quad t \in [0, T).$$

By Lemma 2.7, we let $\xi(t) \in \mathbb{S}$ be a point where this infimum is attained. It follows that

$$m(t) = u_x(t, \xi(t)) \quad \text{and} \quad u_{xx}(t, \xi(t)) = 0.$$

Differentiating the first equation in (2.2) with respect to x and using the identity $\partial_x^2 G * f = G * f - f$, we have

$$u_{tx} + (u - \gamma)u_{xx} = -\frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + u^2 + (\gamma - A)u - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + (\gamma - A)u). \quad (3.1)$$

Since the map $q(t, \cdot)$ given by (2.3) is an increasing diffeomorphism of \mathbb{R} , there exists a $x(t) \in \mathbb{S}$ such that $q(t, x(t)) = \xi(t)$. In particular, $x(0) = \xi(0)$. Note that $u'_0(x_0) = \inf_{x \in \mathbb{S}} u'_0(x)$, we can choose $x_0 = \xi(0)$. It follows that $x(0) = \xi(0) = x_0$. By Lemma 2.3 and the condition $\rho_0(x_0) = 0$, we have

$$\rho(t, \xi(t))q_x(t, x) = \rho(t, q(t, x(t)))q_x(t, x) = \rho_0(x(0)) = \rho_0(x_0) = 0.$$

Thus $\rho(t, \xi(t)) = 0$.

Valuating (3.1) at $(t, \xi(t))$ and using Lemma 2.7, we obtain

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) + \frac{1}{2}u^2 + (\gamma - A)u - (\gamma - A)G * u, \quad (3.2)$$

here we used the relations $G * (u^2 + \frac{1}{2}u_x^2) \geq \frac{1}{2}u^2$ and $G * \rho^2 \geq 0$. Note that $\|G\|_{L^1} = 1$. By Lemma 2.4 and Lemma 2.8, we get

$$\|u\|_{L^\infty}^2 \leq \frac{e+1}{2(e-1)}\|u\|_{H^1}^2 \leq \frac{e+1}{2(e-1)}E_0,$$

$$|(\gamma - A)u| \leq |\gamma - A|\|u\|_{L^\infty} \leq |\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}$$

and

$$|(\gamma - A)G * u| \leq |\gamma - A|\|G\|_{L^1}\|u\|_{L^\infty} \leq |\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}.$$

It follows that

$$\frac{dm(t)}{dt} \leq -\frac{1}{2}m^2(t) + K, \quad (3.3)$$

where $K = \frac{e+1}{4(e-1)}E_0 + 2|\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}}$. Since $m(0) < -\sqrt{2K}$, Lemma 2.10 implies

$$\lim_{t \rightarrow T} m(t) = -\infty \quad \text{with} \quad T = \frac{2u'_0(x_0)}{2K - (u'_0(x_0))^2}.$$

Applying Lemma 2.6, the solution blows up in finite time. \square

Theorem 3.2. Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and T be the maximal time of the solution (u, ρ) to (2.1) with the initial data (u_0, ρ_0) . Assume that $\int_{\mathbb{S}} u_0(x) dx = \frac{a_0}{2}$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and for any $\epsilon > 0$,

$$u'_0(x_0) = \inf_{x \in \mathbb{S}} u'_0(x) < -\sqrt{\frac{\epsilon+2}{24}E_0 + \frac{\epsilon+2}{4\epsilon}a_0^2 + |\gamma - A|\sqrt{\frac{2(\epsilon+2)}{3}E_0 + \frac{4(\epsilon+2)}{\epsilon}a_0^2}},$$

then the corresponding solution to (2.1) blows up in finite time.

Proof. By Lemma 2.5, we have $\int_{\mathbb{S}} u(t, x) dx = \frac{a_0}{2}$. Using Lemma 2.4 and Lemma 2.9, we obtain

$$\|u\|_{L^\infty}^2 \leq \frac{\epsilon + 2}{24} E_0 + \frac{\epsilon + 2}{4\epsilon} a_0^2,$$

$$|(\gamma - A)u| \leq |\gamma - A| \|u\|_{L^\infty} \leq |\gamma - A| \sqrt{\frac{\epsilon + 2}{24} E_0 + \frac{\epsilon + 2}{4\epsilon} a_0^2}$$

and

$$|(\gamma - A)G * u| \leq |\gamma - A| \|G\|_{L^1} \|u\|_{L^\infty} \leq |\gamma - A| \sqrt{\frac{\epsilon + 2}{24} E_0 + \frac{\epsilon + 2}{4\epsilon} a_0^2}.$$

Following the similar proof in Theorem 3.1, we have

$$\frac{dm(t)}{dt} \leq -\frac{1}{2} m^2(t) + K, \quad (3.4)$$

where $K = \frac{\epsilon+2}{48} E_0 + \frac{\epsilon+2}{8\epsilon} a_0^2 + |\gamma - A| \sqrt{\frac{\epsilon+2}{6} E_0 + \frac{\epsilon+2}{\epsilon} a_0^2}$. Following the same argument as in Theorem 3.1, we deduce that the solution blows up in finite time. \square

Letting $a_0 = 0$ and $\epsilon \rightarrow 0$ in Theorem 3.2, we have the following result.

Corollary 3.1. Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and T be the maximal time of the solution (u, ρ) to (2.1) with the initial data (u_0, ρ_0) . Assume that $\int_{\mathbb{S}} u_0(x) dx = 0$. If there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$ and

$$u'_0(x_0) = \inf_{x \in \mathbb{S}} u'_0(x) < -\sqrt{\frac{E_0}{12} + 2|\gamma - A| \sqrt{\frac{E_0}{3}}},$$

then the corresponding solution to (2.1) blows up in finite time.

Remark 3.1. Note that the system (2.1) is variational under the transformation $(u, x) \rightarrow (-u, -x)$ and $(\rho, x) \rightarrow (\rho, -x)$ even $\gamma = 0$. Thus, we can not get a blow up result according to the parity of the initial data (u_0, ρ_0) as we usually do.

Next, we will give more insight into the blow-up mechanism for the wave-breaking solution to the system (2.1), that is the blow-up rate for strong solutions to (2.1).

Theorem 3.3. Let (u, ρ) be the solution to the system (2.1) with the initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, satisfying the assumption of Theorem 3.1, and T be the maximal time of the solution (u, ρ) . Then, we have

$$\lim_{t \rightarrow T} (T - t) \inf_{x \in \mathbb{S}} u_x(t, x) = -2.$$

Proof. As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$.

Define now

$$m(t) := \inf_{x \in \mathbb{S}} [u_x(t, x)], \quad t \in [0, T).$$

By the proof of Theorem 3.1, we have there exists a positive constant $K = K(E_0, \gamma, A)$ such that

$$-K \leq \frac{d}{dt}m + \frac{1}{2}m^2 \leq K \quad \text{a.e. on } (0, T). \quad (3.5)$$

Let $\varepsilon \in (0, \frac{1}{2})$. Since $\liminf_{t \rightarrow T} m(t) = -\infty$ by Theorem 3.1, there is some $t_0 \in (0, T)$ with $m(t_0) < 0$ and $m^2(t_0) > \frac{K}{\varepsilon}$. Since m is locally Lipschitz, it is then inferred from (3.5) that

$$m^2(t) > \frac{K}{\varepsilon}, \quad t \in [t_0, T). \quad (3.6)$$

A combination of (3.5) and (3.6) enables us to infer

$$\frac{1}{2} + \varepsilon \geq -\frac{\frac{dm}{dt}}{m^2} \geq \frac{1}{2} - \varepsilon \quad \text{a.e. on } (0, T). \quad (3.7)$$

Since m is locally Lipschitz on $[0, T)$ and (3.6) holds, it is easy to check that $\frac{1}{m}$ is locally Lipschitz on (t_0, T) . Differentiating the relation $m(t) \cdot \frac{1}{m(t)} = 1$, $t \in (t_0, T)$, we get

$$\frac{d}{dt}\left(\frac{1}{m}\right) = -\frac{\frac{dm}{dt}}{m^2} \quad \text{a.e. on } (t_0, T),$$

with $\frac{1}{m}$ absolutely continuous on (t_0, T) . For $t \in (t_0, T)$. Integrating (3.7) on (t, T) to obtain

$$\left(\frac{1}{2} + \varepsilon\right)(T - t) \geq -\frac{1}{m(t)} \geq \left(\frac{1}{2} - \varepsilon\right)(T - t), \quad t \in (t_0, T),$$

that is,

$$\frac{1}{\frac{1}{2} + \varepsilon} \leq -m(t)(T - t) \leq \frac{1}{\frac{1}{2} - \varepsilon}, \quad t \in (t_0, T).$$

By the arbitrariness of $\varepsilon \in (0, \frac{1}{2})$ the statement of Theorem 3.3 follows. \square

4 Global Existence

In this section, we will present a global existence result.

Theorem 4.1. Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and T be the maximal time of the solution (u, ρ) to (2.1) with the initial data (u_0, ρ_0) . If $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$, then the corresponding solution (u, ρ) exists globally in time.

Proof. Define now

$$m(t) := \inf_{x \in \mathbb{S}} [u_x(t, x)], \quad t \in [0, T).$$

By Lemma 2.7, we let $\xi(t) \in \mathbb{S}$ be a point where this infimum is attained. It follows that

$$m(t) = u_x(t, \xi(t)) \quad \text{and} \quad u_{xx}(t, \xi(t)) = 0.$$

Since the map $q(t, \cdot)$ given by (2.3) is an increasing diffeomorphism of \mathbb{R} , there exists a $x(t) \in \mathbb{S}$ such that $q(t, x(t)) = \xi(t)$.

Set $m(t) = u_x(t, \xi(t)) = u_x(t, q(t, x(t)))$ and $\alpha(t) = \rho(t, \xi(t)) = \rho(t, q(t, x(t)))$. Valuating (3.1) at $(t, \xi(t))$ and using Lemma 2.7, we obtain

$$m'(t) = -\frac{1}{2}m^2(t) + \frac{1}{2}\alpha^2(t) + f \quad \text{and} \quad \alpha'(t) = -m(t)\alpha(t), \quad (4.1)$$

where $f = u^2 + (\gamma - A)u - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + (\gamma - A)u)$. By Lemma 2.4, Lemma 2.8 and $\frac{1}{2\sinh \frac{1}{2}} \leq G(x) \leq \frac{\cosh \frac{1}{2}}{2\sinh \frac{1}{2}}$, we have

$$\begin{aligned} |f| &\leq \|u\|_{L^\infty}^2 + 2|\gamma - A|\|u\|_{L^\infty} + \|G\|_{L^\infty}\|u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\|_{L^1} \\ &\leq \frac{e+1}{2(e-1)}E_0 + 2|\gamma - A|\sqrt{\frac{e+1}{2(e-1)}}E_0^{\frac{1}{2}} + \frac{\cosh \frac{1}{2}}{2\sinh \frac{1}{2}}E_0 := c_1 \end{aligned}$$

By Lemmas 2.2-2.3, we know that $\alpha(t)$ has the same sign with $\alpha(0) = \rho_0(x_0)$ for every $x \in \mathbb{R}$. Moreover, there is a constant $\beta > 0$ such that $|\alpha(0)| = \inf_{x \in \mathbb{S}} |\rho_0(x)| \geq \beta > 0$ because of $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$. Next, we consider the following Lyapunov positive function

$$w(t) = \alpha(0)\alpha(t) + \frac{\alpha(0)}{\alpha(t)}(1 + m^2(t)), \quad t \in [0, T]. \quad (4.2)$$

Letting $t = 0$ in (4.2), we have

$$w(0) \leq \|\rho_0\|_{L^\infty}^2 + 1 + \|u'_0(x)\|_{L^\infty}^2 := c_2.$$

Differentiating (4.2) with respect to t and using (4.1), we obtain

$$\begin{aligned} w'(t) &= \frac{\alpha(0)}{\alpha(t)} \cdot 2m(t)(f + \frac{1}{2}) \\ &\leq \frac{\alpha(0)}{\alpha(t)}(1 + m^2(t))(|f| + \frac{1}{2}) \\ &\leq w(t)(c_1 + \frac{1}{2}). \end{aligned}$$

By Gronwall's inequality, we have

$$w(t) \leq w(0)e^{(c_1 + \frac{1}{2})t} \leq c_2e^{(c_1 + \frac{1}{2})t}$$

for all $t \in [0, T]$. On the other hand,

$$w(t) \geq 2\sqrt{\alpha^2(0)(1 + m^2(t))} \geq 2\beta|m(t)|, \quad \forall t \in [0, T].$$

Thus,

$$|m(t)| \leq \frac{1}{2\beta}w(t) \leq \frac{c_2}{2\beta}e^{(c_1 + \frac{1}{2})t}$$

for all $t \in [0, T]$. It follows that

$$\liminf_{t \rightarrow T} m(t) \geq -\frac{c_2}{2\beta}e^{(c_1 + \frac{1}{2})T}.$$

This completes the proof by using Lemma 2.6.

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